4.2 Nilpotent Lie algebras and Lie groups.

Nilpotent Lie algebras and Lie groups play a fundamental role in the structure theory.

Our focus in this section will be on nilpotent Lie algebras. We will shortly indicate the relationship with nilpotent Lie groups at the end.

Most proofs will be omitted. Often they are analogous to the proofs of the "corresponding" statements in the solvable case.

Let $\mathfrak{g}$ be a Lie algebra. Let $\mathfrak{a}$ and $\mathfrak{b}$ be subspaces of $\mathfrak{g}$. Then

$$[\mathfrak{a},\mathfrak{b}] := \text{Linear Span}\{[X,Y] : X \in \mathfrak{a}, Y \in \mathfrak{b}\}.$$

$\boxed{\text{Definition 4.33}}$ [Nilpotent Lie algebra]

A Lie algebra $\mathfrak{g}$ is said to be nilpotent if there is a sequence

$$\mathfrak{g} = \mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \cdots \supset \mathfrak{g}_r = \{0\}$$

of subspaces such that $[\mathfrak{g}, \mathfrak{g}_i] \subset \mathfrak{g}_{i-1}$, $1 \leq i \leq r$.

Note: clearly the $\mathfrak{g}_i$'s must be ideals in $\mathfrak{g}$. Moreover, if $\pi : \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{g}_i$ is the canonical projection we have

check it!

$$[\mathfrak{g}/\mathfrak{g}_i, \mathfrak{g}_{i-1}/\mathfrak{g}_i] = \pi([\mathfrak{g}, \mathfrak{g}_{i-1}]) \subset \pi(\mathfrak{g}_i) = \{0\}$$

Hence $\mathfrak{g}_{i-1}/\mathfrak{g}_i \subset Z(\mathfrak{g}/\mathfrak{g}_i)$.

As in the case of solvable Lie algebras we define, for any Lie algebra $\mathfrak{g}$,

$$C^1(\mathfrak{g}) = \mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}]$$

and inductively $C^i(\mathfrak{g}) = [\mathfrak{g}, C^{i-1}(\mathfrak{g})]$ $i \geq 2$.

Definition 4.34 [Central series]

$C^{\ell}(\mathfrak{g})$, $\ell \geq 1$ is called the central series of $\mathfrak{g}$.

A key property of the central series is the following:

$$(*) \qquad C^{i-1}(\mathfrak{g}) / C^{i}(\mathfrak{g}) \subset Z\left(\mathfrak{g} / C^{i}(\mathfrak{g})\right).$$

Proposition 4.35

The following are equivalent:

1) $\mathfrak{g}$ is nilpotent;

2) $C^{r}(\mathfrak{g}) = 0$ for some $r \geq 1$;

3). $\exists\, m \geq 1$ s.t.

$$\operatorname{ad}(X_1) \circ \cdots \circ \operatorname{ad}(X_m) = 0$$

$\forall\, X_1, \ldots, X_m \in \mathfrak{g}$.

Sketch of proof

2) $\Leftrightarrow$ 3) follows immediately from the observation that

$C^k(\mathfrak{g})$ is the linear span of
$\mathrm{ad}(X_1)\circ \ldots \circ \mathrm{ad}(X_r)(Y)$ for
$X_1, \ldots, X_r, Y \in \mathfrak{g}$. $\square$

Definition 4.36 [Nilpotency length]

If $\mathfrak{g}$ is a nilpotent Lie algebra then:

$$\mathrm{nil}(\mathfrak{g}) := \min\{r \geq 1 : C^r(\mathfrak{g}) = \{0\}\}.$$

Example 4.37

1) Any nilpotent Lie algebra is solvable.
This follows from the inclusions

$$C^k(\mathfrak{g}) \supset \mathfrak{g}^{(k)} \quad \text{for all } k \geq 1.$$

2) $\mathfrak{n} = \left\{ \begin{pmatrix} 0 & \ddots & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix} : * \in \mathbb{R} \right\} \subseteq \mathfrak{gl}(n, \mathbb{R})$

is a nilpotent Lie algebra.

Indeed:

$$C^1(\mathfrak{n}) = \left\{ \begin{pmatrix} 0 & 0 & \ddots & * \\ & \ddots & \ddots & 0 \\ 0 & & & 0 \end{pmatrix} : * \in \mathbb{R} \right\}$$

and $C^n(n) = \{0\}$.

$\circ$

Although there are solvable Lie algebras that are not nilpotent we have:

Theorem 4.38

A Lie algebra $\mathfrak{g}$ is solvable iff. $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent.

Remark 4.39

Let $\mathfrak{g}$ be a nilpotent Lie algebra with $nil(\mathfrak{g}) = n$. Then $C^n(\mathfrak{g}) = \{0\}$ and

$$0 \neq C^{n-1}(\mathfrak{g}) \subset Z(\mathfrak{g})$$

thanks to the key property $(*)$ above.

In particular, if $\mathfrak{g}$ is nilpotent and $\mathfrak{g} \neq \{0\}$ then $Z(\mathfrak{g}) \neq \{0\}$.

Key to the proof of Theorem 4.39 is the following lemma of independent interest:

Lemma 4.40

Let $\mathfrak{h} \triangleleft \mathfrak{g}$ be an ideal.

1) $\mathfrak{g}$ nilpotent $\implies$ $\mathfrak{h}$ and $\mathfrak{g}/\mathfrak{h}$ are nilpotent.

2) If $\mathfrak{g}/\mathfrak{h}$ is nilpotent and $\mathfrak{h} \subset Z(\mathfrak{g})$ then $\mathfrak{g}$ is nilpotent.

$\boxed{\text{Remark 4.41}}$

Let $\mathfrak{g} = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} : * \in \mathbb{R} \right\}$ and

$$\mathfrak{h} = \left\{ \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix} : * \in \mathbb{R} \right\}.$$

Then $\mathfrak{g}/\mathfrak{h}$ and $\mathfrak{h}$ are abelian, in particular, they are nilpotent. However, $\mathfrak{g}$ is not nilpotent since $Z(\mathfrak{g}) = \{0\}$.

<u>Proof of Theorem 4.39</u>

We only discuss the implication $(\Leftarrow)$.

Note that $\mathfrak{g}^{(r+1)} = (\mathfrak{g}^{(1)})^{(r)} \subset C^r(\mathfrak{g}^{(1)})$.

Hence if $\mathfrak{g}^{(1)}$ is nilpotent $\mathfrak{g}$ is solvable. $\square$

We next turn to Engel's theorem which is the analogue of Lie's theorem (Theorem 4.32) for solvable Lie algebras.

Example 4.42

Consider:

$$\mathfrak{n} = \left\{ \begin{pmatrix} 0 & & x \\ & \ddots & \\ 0 & & 0 \end{pmatrix} : x \in \mathbb{R} \right\} \quad \text{and}$$

$$\mathfrak{a} = \left\{ \begin{pmatrix} x & & 0 \\ & \ddots & \\ 0 & & x \end{pmatrix} : x \in \mathbb{R} \right\}.$$

Both $\mathfrak{n}$ and $\mathfrak{a}$ are nilpotent Lie algebras. However, while $\mathfrak{n}$ is in strictly upper triangular form there is no change of basis that would make $\mathfrak{a}$ strictly upper triangular.

Theorem 4.43 [Engel's theorem]

Let $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ be a representation of a Lie algebra $\mathfrak{g}$ into a $\mathbb{R}$-vector space such that $\rho(X)$ is nilpotent $\forall X \in \mathfrak{g}$. Then there is a basis of V with respect to which $\rho(\mathfrak{g})$ is strictly upper triangular.

Note: $\rho(x) \in gl(V)$ being nilpotent means that there exists $n \in \mathbb{N}$ such that $(\rho(x))^n = 0$.

We omit the proof of Theorem 4.43 and discuss an important corollary:

$\boxed{\text{Corollary 4.44}}$

A Lie algebra $\mathfrak{g}$ is nilpotent iff $ad(\mathfrak{g})$ is strictly upper triangular with respect to some basis.

<u>Proof</u>

$(\Leftarrow)$ If there is a basis of $\mathfrak{g}$ such that:

$$ad(\mathfrak{g}) \subset n = \left\{ \begin{pmatrix} 0 & * & * \\ & \ddots & * \\ 0 & & 0 \end{pmatrix} , * \in \mathbb{R} \right\}$$

then $ad(\mathfrak{g})$ is nilpotent. In addition $\operatorname{Ker} ad = Z(\mathfrak{g})$ by Corollary 3.106.

Hence by Lemma 4.40 2) $\mathfrak{g}$ is nilpotent.

($\Leftarrow$) If $\mathfrak{g}$ is nilpotent then by Proposition 4.35 3) $\mathrm{ad}(x)$ is nilpotent $\forall x \in \mathfrak{g}$. The conclusion follows from Theorem 4.43.

$\square$

We end this section with a brief discussion about nilpotent (Lie) groups.

Definition 4.45

Given a group G and subgroups $A, B < G$ we define

$[A,B] :=$ subgroup of G generated by $\{[a,b] : a \in A, b \in B\}$.

Remark 4.46

If $A \triangleleft G$ and $B \triangleleft G$ then $[A,B] \triangleleft G$.

Definition 4.47

A group G is said to be nilpotent if there is a sequence

$$G = G_0 \trianglerighteq G_1 \trianglerighteq \cdots \trianglerighteq G_r = \{e\}$$

with $[G, G_{i-1}] \leq G_i \quad 1 \leq i \leq r$.

Clearly the above condition implies that $G_{i-1} \trianglelefteq G$ $1 \leq i \leq r$. Moreover, the condition $[G, G_{i-1}] \subset G_i$ is equivalent to

$$G_{i-1}/G_i \subset Z(G/G_i).$$

As in the case of Lie algebras we can define inductively

$$C^1(G) := [G, G],$$
$$C^i(G) := [G, C^{i-1}(G)] \quad i \geq 2.$$

Definition 4.48 [Descending central series]
$C^i(G)$ is the descending central series of G.

Lemma 4.49
G is nilpotent iff there is $r \geq 1$ s.t. $C^r(G) = \{e\}$.

With the very same method as in the proof of Theorem 4.29 for solvable groups one can prove:

Theorem 4.50

Let G be a connected Lie group. Then the following are equivalent:

1) G is nilpotent;

2) There is a sequence of closed connected subgroups

$$G = G_0 > G_1 > \cdots > G_r = \{e\}$$

with $[G, G_{i-1}] \subset G_i$, $1 \le i \le r$;

3) $\mathfrak{g} = \mathrm{Lie}(G)$ is a nilpotent Lie algebra.

4.3 The Killing form and Cartan's criterion for solvability.

In this short section we briefly introduce the Killing form and discuss its role in the study of solvable Lie algebras via Cartan's criterion.

From now on $\mathfrak{g}$ will be a $\mathbb{K}$ Lie algebra with $\mathbb{K}=\mathbb{R}$ or $\mathbb{K}=\mathbb{C}$.

Definition 4.51 [Killing form]
The Killing form of a $\mathbb{K}$-Lie algebra is the bilinear form

$$K_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{R}.$$

defined by

$$K_{\mathfrak{g}}(x, y) := \operatorname{tr}(\operatorname{ad}(x)\operatorname{ad}(y)).$$

The following invariance property is key for the applications:

Proposition 4.52.

$$K_{\mathfrak{g}}(\mathrm{ad}(Z)X, Y) + K_{\mathfrak{g}}(X, \mathrm{ad}(Z)Y) = 0$$
$\forall X, Y, Z \in \mathfrak{g}$

Proof

Recall $\mathrm{ad}(Z)X = [Z, X]$, $\mathrm{ad}(Z)Y = [Z, Y]$.

$$K_{\mathfrak{g}}(\mathrm{ad}(Z)X, Y) + K_{\mathfrak{g}}(X, \mathrm{ad}(Z)Y) =$$
$$= \mathrm{tr}(\mathrm{ad}([Z,X])\,\mathrm{ad}(Y)) + \mathrm{tr}(\mathrm{ad}(X)\,\mathrm{ad}([Z,Y]))$$
$$= \mathrm{tr}([\mathrm{ad}(Z), \mathrm{ad}(X)]\,\mathrm{ad}(Y)) + \mathrm{tr}(\mathrm{ad}(X)[\mathrm{ad}(Z), \mathrm{ad}(Y)])$$

$$= \mathrm{tr}(\mathrm{ad}(Z)\,\mathrm{ad}(X)\,\mathrm{ad}(Y)) - \mathrm{tr}(\mathrm{ad}(X)\,\mathrm{ad}(Z)\,\mathrm{ad}(Y))$$
$$+ \mathrm{tr}(\mathrm{ad}(X)\,\mathrm{ad}(Z)\,\mathrm{ad}(Y)) - \mathrm{tr}(\mathrm{ad}(X)\,\mathrm{ad}(Y)\,\mathrm{ad}(Z))$$

$$= \mathrm{tr}(\mathrm{ad}(Z)\,\mathrm{ad}(X)\,\mathrm{ad}(Y)) - \mathrm{tr}(\mathrm{ad}(X)\,\mathrm{ad}(Y)\,\mathrm{ad}(Z))$$

$\rightsquigarrow$ $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.

$= 0$. $\square$

$\boxed{\text{Exercise 4.53}}$

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Prove that:

$$K_{\mathfrak{g}}(\mathrm{Ad}(g)X, \mathrm{Ad}(g)Y) = K_{\mathfrak{g}}(X, Y)$$

for all $g \in G$ and $X, Y \in \mathfrak{g}$.

Hint: compute the derivative with respect to t of

$K_g(\mathrm{Ad}(\exp tZ)X, \mathrm{Ad}(\exp tZ)Y)$.

Theorem 4.54 [Cartan's criterion]

A $\mathbb{k}$-Lie algebra is solvable if and only if $K_g|_{g^{(1)} \times g^{(1)}} = 0$.

We discuss only one implication. The following lemma will be important for us:

Lemma 4.55

Let $h \triangleleft g$ be an ideal. Then $K_g|_{h\times h} = K_h$.

Proof

Let V be a linear complement of h in g, so that $g = h \oplus V$.

If we consider $\mathrm{ad}_g(X) : h \oplus V \longrightarrow h \oplus V$ then:

$\mathrm{ad}_g(X)Y = [X, Y] \in h$ if $Y \in h$

$\mathrm{ad}_g(X)Y = [X, Y] \in h$ if $Y \in V$

since $\mathfrak{h}$ is an ideal. Note that being a subalgebra would be sufficient for the first conclusion.

Hence $ad_{\mathfrak{g}}(x)$ can be represented as

$$\begin{array}{ccc} \mathrm{ad}_{\mathfrak{g}}(x) = \begin{pmatrix} \mathrm{ad}_{\mathfrak{h}}(x) & * \\ 0 & 0 \end{pmatrix} & \begin{array}{c} \mathfrak{h} \\ V \end{array} \\ \quad\quad\quad\quad \mathfrak{h} \quad\quad\quad V & \end{array}$$

Therefore,

$$\begin{aligned} K_{\mathfrak{g}}(x,y) &= tr(ad_{\mathfrak{g}}(x) ad_{\mathfrak{g}}(y)) \\ &= tr(ad_{\mathfrak{h}}(x)\, ad_{\mathfrak{h}}(y)) \\ &= K_{\mathfrak{h}}(x,y) \end{aligned}$$

for all $x, y \in \mathfrak{h}$. $\square$

Proof of ($\Rightarrow$) in Theorem 4.54

Assume that $\mathfrak{g}$ is solvable. Then by Theorem 4.38 $\mathfrak{g}^{(1)} = [\mathfrak{g},\mathfrak{g}]$ is nilpotent. By Corollary 4.44 $ad(\mathfrak{g}^{(1)})$ is strictly upper triangular with respect to some basis. Taking into account Lemma 4.55 above $K_{\mathfrak{g}}|_{\mathfrak{g}^{(1)} \times \mathfrak{g}^{(1)}} = K_{\mathfrak{g}^{(1)}} = 0$ since $\mathfrak{g}^{(1)} \triangleleft \mathfrak{g}$. $\square$

4.4 Semisimple Lie algebras and Lie groups.

Let as before $\mathfrak{g}$ be a Lie algebra over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

Definition 4.56

1) $\mathfrak{g}$ is simple if
   a) $\mathfrak{g}$ is non-abelian;
   b) if $\mathfrak{h} \trianglelefteq \mathfrak{g}$ then either $\mathfrak{h} = \{0\}$ or $\mathfrak{h} = \mathfrak{g}$.

2) $\mathfrak{g}$ is semisimple if there are simple ideals $\mathfrak{h}_1, \ldots, \mathfrak{h}_n$ in $\mathfrak{g}$ such that as Lie algebras.

$$\mathfrak{g} = \mathfrak{h}_1 \oplus \ldots \oplus \mathfrak{h}_r .$$

This means that if $X = \sum_{i=1}^{r} X_i$ and $Y = \sum_{i=1}^{r} Y_i$ with $X_i, Y_i \in \mathfrak{h}_i$ then

$$[X, Y] = \sum_{i=1}^{r} [X_i, Y_i] .$$

3) A connected Lie group is simple (respectively semisimple) if its

Lie algebra is.

Remark 4.57

An abstract group G is simple if it admits only two normal subgroups G itself and $\{e\}$.
We shall see that $SL(n,\mathbb{R})$ is a simple Lie group. However, it is not simple as an abstract group since $Z(SL(n,\mathbb{R})) = \{\pm 1\!\!1\}$.

The fundamental characterization of semisimplicity is given by the following:

Theorem 4.58

$\mathfrak{g}$ is semisimple if and only if $K_{\mathfrak{g}}$ is non-degenerate.

———— o ————

Recall: a symmetric bilinear form $C : V \times V \longrightarrow \mathbb{K}$ is said to be non-degenerate if, setting
$\mathrm{rad}(C) := \{ v \in V : C(u,v) = 0 \ \forall u \in V \}$

it holds $\mathrm{rad}(C) = \{0\}$.

$$\text{———} \circ \text{———}$$

We will discuss one implication in the proof. For this we need the following:

Lemma 4.59

Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{h} \trianglelefteq \mathfrak{g}$ be an ideal. Then

$$\mathfrak{h}^{\perp} := \{x \in \mathfrak{g} : \kappa_{\mathfrak{g}}(x,y) = 0 \ \forall\, y \in \mathfrak{h}\}$$

is an ideal as well.

Proof

Let $z \in \mathfrak{g}$, $x \in \mathfrak{h}^{\perp}$, $y \in \mathfrak{h}$. Then

$$\kappa_{\mathfrak{g}}(\mathrm{ad}(z)(x), y) = -\kappa_{\mathfrak{g}}(x, \mathrm{ad}(z)y) = 0$$

where we used Proposition 4.52 for the first equality and the assumption that $\mathfrak{h}$ is an ideal for the second one. $\square$

Proof of ($\Longrightarrow$) in Theorem 4.58

Assume that $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r$ with $\mathfrak{g}_i$ simple for $1 \le i \le r$.

Then $\forall X = \sum_{i=1}^{r} X_i$ it holds

$$\mathrm{ad}_{\mathfrak{g}}(X) = \begin{pmatrix} \mathrm{ad}_{\mathfrak{g}_1}(X_1) & & \\ & \ddots & \\ & & \mathrm{ad}_{\mathfrak{g}_r}(X_r) \end{pmatrix}.$$

Hence $K_{\mathfrak{g}}(X,Y) = \sum_{i=1}^{r} K_{\mathfrak{g}_i}(X_i, Y_i) \ \forall X,Y \in \mathfrak{g}$.

Therefore it is sufficient to discuss the case when $\mathfrak{g}$ is simple (Check it!).

Let $\mathfrak{g}^{\perp} = \mathrm{rad}(K_{\mathfrak{g}}) = \{ Y \in \mathfrak{g} : K_{\mathfrak{g}}(X,Y) = 0 \ \forall X \in \mathfrak{g} \}$

Then $\mathfrak{g}^{\perp}$ is an ideal in $\mathfrak{g}$ by Lemma 4.59.
Since $\mathfrak{g}$ is simple, either $\mathfrak{g}^{\perp} = \{0\}$ or $\mathfrak{g}^{\perp} = \mathfrak{g}$. If $\mathfrak{g}^{\perp} = \mathfrak{g}$ then $K_{\mathfrak{g}} \equiv 0$ hence $\mathfrak{g}$ is solvable by Cartan's Theorem 4.54, a contradiction to simplicity.
Hence $\mathfrak{g}^{\perp} = \{0\}$, i.e., $K_{\mathfrak{g}}$ is non-degenerate

$\square$

Next we discuss a powerful way to produce families of semisimple Lie algebras:

Theorem 4.60

Let V be a $\mathbb{K}$-vector space endowed with an inner product $\langle\, , \rangle$.
If $\mathfrak{g} \subset \mathfrak{gl}(V)$ is a $\mathbb{K}$-subalgebra that is self-adjoint under $\langle\, , \rangle$ and such that $Z(\mathfrak{g}) = \{0\}$ then $K_{\mathfrak{g}}$ is non-degenerate and hence $\mathfrak{g}$ is semisimple.

Note: for $A \in \mathfrak{gl}(V)$ we let $A^* \in \mathfrak{gl}(V)$ be defined by

$$\langle Av, w \rangle = \langle v, A^* w \rangle \quad \forall v, w \in V.$$

Then $\mathfrak{g}$ being self-adjoint means that $\forall A \in \mathfrak{g}$ it holds $A^* \in \mathfrak{g}$.

We can exploit Theorem 4.60 to produce a large family of examples of semisimple Lie algebras.

Example 4.61

1) $sl(n,\mathbb{R}) \subset gl(n,\mathbb{R})$ is invariant under $A \longmapsto {}^tA$ and $Z(sl(n,\mathbb{R})) = \{0\}$.

2) $sl(n,\mathbb{C}) \subset gl(n,\mathbb{C})$ is invariant under $A \longmapsto {}^t\bar{A}$ and $Z(sl(n,\mathbb{C})) = \{0\}$.

3) For $p+q=n$,

$$o(p,q) := \{ X \in gl(n,\mathbb{R}) : {}^tX J_{p,q} + J_{p,q} X = 0 \}$$

where $J_{p,q} = \begin{pmatrix} \mathrm{Id}_p & 0 \\ 0 & -\mathrm{Id}_q \end{pmatrix}$ is invariant under $X \longmapsto {}^tX$. Indeed from ${}^tX J_{p,q} + J_{p,q} X = 0$ we obtain by multiplying on the left and on the right by $J_{p,q}$ that $J_{p,q}\,{}^tX + X J_{p,q} = 0$ since $J_{p,q}^2 = \mathrm{Id}_n$.

One can also verify that $Z(o(p,q)) = \{0\}$.

—— o ——

We conclude with some hints towards the so called Levi decomposition of Lie groups.

Proposition 4.62

For any Lie algebra $\mathfrak{g}$ there is a unique maximal solvable ideal $\mathfrak{r} \subset \mathfrak{g}$. Moreover $\mathfrak{g}/\mathfrak{r}$ is semisimple.

Definition 4.63 [Radical]

The unique maximal solvable ideal from Proposition 4.62 is called the (solvable) radical of $\mathfrak{g}$.

For the proof of Proposition 4.62 we need the following:

Lemma 4.64

If $\mathfrak{a}$ and $\mathfrak{b}$ are solvable ideals in a Lie algebra $\mathfrak{g}$ then $\mathfrak{a}+\mathfrak{b}$ is a solvable ideal.

The proof of Lemma 4.64 is left as an exercise. We discuss how to use it to prove Proposition 4.62.

Proof of Proposition 4.62

In order to prove existence and uniqueness of the maximal solvable ideal it suffices to exploit finite dimensionality and Lemma 4.64.

To show that $\mathfrak{g}/\mathfrak{r}$ is semisimple we first establish the following:
Claim: $\mathfrak{g}/\mathfrak{r}$ has no solvable ideals.

Indeed, let $\pi: \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{r}$ be the canonical projection and let $\mathfrak{h} \subset \mathfrak{g}/\mathfrak{r}$ be a solvable ideal. Then $\mathfrak{a} := \pi^{-1}(\mathfrak{h}) \subseteq \mathfrak{g}$ is an ideal. Moreover $\pi(\mathfrak{a}) = \mathfrak{h}$ is solvable and $\operatorname{Ker} \pi|_{\mathfrak{a}}$ is solvable being contained in $\mathfrak{r}$. Therefore $\mathfrak{a}$ is solvable and hence $\mathfrak{a} \subseteq \mathfrak{r}$ and $\mathfrak{h} = \{0\}$.

Next it is sufficient to observe that a Lie algebra with no non-trivial solvable ideals is semisimple.

The proof is left as an Exercise. $\square$

Hint: prove and then use the following

$\boxed{\text{Proposition 4.65}}$

Let $\mathfrak{g} = \bigoplus_I \mathfrak{g}_i$ be a direct sum of simple ideals. Then any ideal $\mathfrak{h} \trianglelefteq \mathfrak{g}$ is of the form $\mathfrak{h} = \bigoplus_{i\in J} \mathfrak{g}_i$ where $J \subseteq I$.